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Stability of Stationary Solutions of a Semilinear Parabolic Partial Differential Equation

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We consider a parabolic partial differential equation $u_t = u_{xx} + f(u)$ on a compact interval of spatial variable x with Dirichlet boundary conditions. The stability of stationary solutions of this system is studied by the use of Liapunov's second method. We obtain necessary and sufficient conditions for the stability, asymptotic stability, neutral stability, instability, and conditional stability. These conditions are closely connected with the conditions for the existence of the stationary solutions.

1. INTRODUCTION

In this paper we consider the following parabolic partial differential equation

$$u_t(x, t) = u_{xx}(x, t) + f(u(x, t)), \qquad 0 < x < l. \tag{1.1a}$$

Here, the nonlinear term $f(u)$ is assumed to be a sufficiently smooth function. Equations of type (1.1a) with appropriate boundary conditions appear in various fields such as chemical reactor dynamics and nonlinear heat transfer [4, 5]. In this paper we assume the Dirichlet boundary conditions

$$u(0, t) = u_1, \qquad u(l, t) = u_2. \tag{1.1b}$$

Without losing any generality, we shall assume that $u_2 \geqslant u_1$.

We study the stability of stationary solutions of this system. In recent years, applications of Liapunov's second method to the stability analysis of partial differential equations have been developed by many authors including [6–12]. Our study is in the same direction.

In Section 2 we shall obtain necessary and sufficient conditions which u_1, u_2 and $f(u)$ must satisfy for the existence of stationary solutions of various types. In Section 3, we shall obtain several criteria for determining the stability of these stationary solutions (Theorems 3.1–3.7). By the use of these criteria, we can determine, given a stationary solution, whether it is stable or unstable. Also,

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in the case it is stable, we can determine whether it is asymptotically stable or neutrally stable. In certain cases where a stationary solution is unstable, it may be stable with respect to a suitably restricted class of disturbances. In such cases, we say that it is conditionary stable. We shall obtain conditions under which a stationary solution is conditionally stable with respect to what we shall call positive or negative disturbances. Our stability criteria given in Theorems 3.1–3.7 are closely connected with the conditions for the existence of the stationary solutions.

Our theorems regarding the stability are proved in Sections 4 and 5. The main part of the stability analysis is carried out in Section 5 by the use of Liapunov's second method.

It is very difficult to determine the stability of a stationary solution when the linearized perturbation equation is neutrally stable. In such a case, we must consider the effect of higher-order terms which can be neglected in the ordinary case. Our stability analysis gives a definite answer to this problem.

The method of stability analysis developed in this paper can be extended and effectively used in studying the waveform stability of traveling wave solutions which appear in the Gunn-diode equation [13]. We shall treat this problem in a subsequent paper [14].

2. Existence of Stationary Solutions

A function $u_0(x)$ is a stationary solution of (1.1) if and only if it is twice continuously differentiable and satisfies the equation

$$u_{0xx}(x) + f(u_0(x)) = 0, \qquad 0 < x < l, \tag{2.1a}$$

with the boundary conditions

$$u_0(0) = u_1\,, \qquad u_0(l) = u_2\,. \tag{2.1b}$$

In order to avoid repetition of arguments, we assume that $u_{0x}(0) \geqslant 0$. We shall obtain necessary and sufficient conditions for the existence of $u_0(x)$.

Let $F(u)$ be any primitive function of $f(u)$. For a real variable D limited in the range

$$D \geqslant \operatorname*{Max}_{u_1 \leqslant u \leqslant u_2} F(u) \equiv F_0\,, \tag{2.2}$$

let us define a function $x_1(D)$ by

$$x_1(D) \equiv \int_{u_1}^{u_2} \frac{dw}{(2\{D - F(w)\})^{1/2}}\,.$$

Also, for any D in the range (2.2) let us define $w(x; D)$ by the equation

$$w_{xx}(x; D) + f(w(x; D)) = 0, \tag{2.3a}$$

with the initial conditions

$$w(0; D) = u_1, \qquad w_x(0; D) = (2\{D - F(u_1)\})^{1/2}. \tag{2.3b}$$

Clearly a solution $w(x; D)$ of (2.3) satisfies

$$\tfrac{1}{2}\{w_x(x; D)\}^2 + F(w(x; D)) = D. \tag{2.4}$$

Hence we can apply the phase-energy method [11] to obtain that

$$w_x(x; D) > 0 \qquad \textit{for all } x \textit{ in} \quad [0, x_1(D)],$$
$$w(x_1(D); D) = u_2.$$

Thus the following theorem holds.

THEOREM 2.1. *There exists a stationary solution $u_0(x)$ which satisfies $u_{0x}(x) > 0$ for all x in $[0, l]$ if and only if the equation $x_1(D) = l$ has a real solution $D = \bar{D}$ in the range* (2.2). *The stationary solution is given by $u_0(x) = w(x; \bar{D})$.*

We define F_1 and F_2, respectively, by

$$F_1 \equiv \sup_{u \leqslant u_1} F(u), \qquad F_2 = \sup_{u \geqslant u_2} F(u). \tag{2.5}$$

Let us consider the case where $F_2 \geqslant F_0$. In this case, for any D lying in $[F_0, F_2]$, we can define a function $a(D) \geqslant u_2$ by

$$F(a(D)) = D, \qquad F(u) \leqslant D \qquad \textit{for all } u \textit{ in} \quad [u_2, a(D)]. \tag{2.6}$$

Let us define functions $x_2(D)$ and $S(D)$ by

$$x_2(D) \equiv \int_{u_1}^{a(D)} \frac{dw}{(2\{D - F(w)\})^{1/2}}, \qquad S(D) \equiv x_2(D) + \int_{u_2}^{a(D)} \frac{dw}{(2\{D - F(w)\})^{1/2}}. \tag{2.7}$$

It is easily verified by the use of phase-energy method that

$$\begin{aligned} w_x(x; D) &> 0 && \text{for all } x \quad \text{in} \quad (0, x_2(D)), \\ w(x_2(X); D) &= a(D), && w_x(x_2(D); D) = 0, \\ w_x(x; D) &< 0 && \text{for all } x \quad \text{in} \quad (x_2(D), S(D)), \\ w(S(D); D) &= u_2. \end{aligned} \tag{2.8}$$

(See Fig. 1.) Hence the following theorem holds.

THEOREM 2.2. *There exists a stationary solution $u_0(x)$ which has a peak at a certain point x_2 in $(0, l)$ and satisfies $u_{0x}(x) \neq 0$ for all x in $(0, x_2) \cup (x_2, l)$ if and*

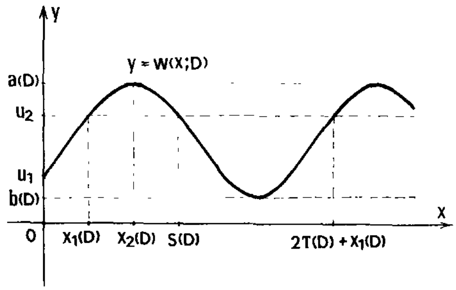

FIG. 1. The function $w(x; D)$.

only if the equation $S(D) = l$ has a real solution $D = D^*$ in the range $[F_0, F_2]$. The stationary solution is given by $u_0(x) = w(x; D^*)$.

Let us consider the case where $F_1 \geqslant F_0$ and $F_2 \geqslant F_0$. In this case, for any D lying in the range

$$F_0 \leqslant D \leqslant \operatorname{Min}\{F_1, F_2\}, \tag{2.9}$$

we can define functions $b(D) \leqslant u_1$ and $T(D)$, respectively, by

$$F(b(D)) = D, \qquad F(u) \leqslant D \qquad \text{for all } u \text{ in } \quad [b(D), u_1], \tag{2.10}$$

$$T(D) \equiv \int_{b(D)}^{a(D)} \frac{dw}{(2\{D - F(w)\})^{1/2}}. \tag{2.11}$$

If D lies in the range (2.9), then $w(x; D)$ is a periodic function of x with the minimal period $2T(D)$. Hence, if there exists a $D = D^*$ satisfying either $2kT(D^*) + S(D^*) = l$ or $2kT(D^*) + x_1(D^*) = l$ for some integer $k \geqslant 0$, then $w(x; D^*)$ satisfies the boundary condition (2.1b), i.e., $u_0(x) = w(x; D^*)$ is a stationary solution. If $k \geqslant 1$, this stationary solution satisfies $u_{0x}(x) = 0$ at two or more points in $(0, l)$.

We denote by $\hat{w}(x; D)$ a solution of Eq. (2.3a) subject to the initial condition

$$\hat{w}(0; D) = b(D), \qquad \hat{w}_x(0; D) = 0. \tag{2.12}$$

If D is limited in the range (2.9), $\hat{w}(x; D)$ satisfies

$$\begin{aligned} &\hat{w}_x(x; D) > 0 && \text{for all } x \quad \text{in} \quad (0, T(D)), \\ &\hat{w}(T(D); D) = a(D), && \hat{w}_x(T(D); D) = 0. \end{aligned} \tag{2.13}$$

Let us consider a stationary solution $u_0(x)$ which satisfies, in addition to the boundary condition (2.1b), the condition

$$\begin{aligned} &u_{0x}(0) = u_{0x}(l) = 0, \\ &u_{0x}(x) > 0 \qquad \text{for all } x \quad \text{in} \quad (0, l). \end{aligned} \tag{2.14}$$

By virtue of (2.12) and (2.13), there exists a stationary solution of the type (2.14) if and only if there exists a $D = D^*$ in the range (2.9) such that

$$a(D^*) = u_2, \qquad b(D^*) = u_1, \qquad T(D^*) = l. \tag{2.15}$$

If (2.15) holds, a stationary solution of the type (2.14) is given by $u_0(x) = \hat{w}(x; D^*)$

3. Stability of Stationary Solutions

3.1. *Definition of Stability*

Let $u_0(x)$ be a stationary solution of (1.1) and $v_0(x)$ be any continuously differentiable function defined on $[0, l]$ satisfying

$$v_0(0) = v_0(l) = 0. \tag{3.1}$$

We denote by $u(x, t)$ a solution of Eq. (1.1) subject to the initial condition

$$u(x, 0) = u_0(x) + v_0(x), \qquad 0 \leqslant x \leqslant l. \tag{3.2}$$

The stationary solution $u_0(x)$ is said to be *stable* if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that, for any $v_0(x)$ satisfying (3.1) and

$$\operatorname*{Max}_{0 \leqslant x \leqslant l} |\, v_0(x)| < \delta, \qquad |\, v_{0x}(0)| + |\, v_{0x}(l)| < \delta, \tag{3.3}$$

the solution $u(x, t)$ satisfies $\operatorname{Max}_{0 \leqslant x \leqslant l} |\, u(x, t) - u_0(x)| < \epsilon$ for all $t \geqslant 0$. We say that $u_0(x)$ is *asymptotically stable* if, in addition to being stable, $u(x, t)$ satisfies

$$\lim_{t \to \infty} \operatorname*{Max}_{0 \leqslant x \leqslant l} |\, u(x, t) - u_0(x)| = 0 \tag{3.4}$$

when $\delta > 0$ in (3.3) is sufficiently small. Also, we say that $u_0(x)$ is *neutrally stable* if, in addition to being stable, $u(x, t)$ converges, as $t \to \infty$, to a certain stationary solution which is not always equal to $u_0(x)$.

In certain cases where the stationary solution $u_0(x)$ is unstable, the solution may be stable when the disturbance $v_0(x)$ is limited to a suitably restricted class. We say that $u_0(x)$ is *stable with respect to positive* [*negative*] *disturbances* if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that, for any $v_0(x)$ satisfying (3.1), (3.3), and

$$v_0(x) \geqslant 0 \ [v_0(x) \leqslant 0] \qquad \textit{for all } x \textit{ in} \quad [0, l], \tag{3.5}$$

the solution $u(x, t)$ satisfies

$$\epsilon > u(x, t) - u_0(x) \geqslant 0, \qquad [-\epsilon < u(x, t) - u_0(x) \leqslant 0]$$

for all (x, t) in $[0, l] \times [0, \infty)$. Also, we say that $u_0(x)$ is *asymptotically stable with respect to positive* [*negative*] *disturbances* if $u(x, t)$ satisfies (3.4) for any $v_0(x)$ satisfying (3.1), (3.3), and (3.5), and for a sufficiently small $\delta > 0$ in (3.3).

3.2. *Main Theorems*

We have several theorems regarding the stability of the stationary solutions. These are the main results of this paper. The first and the second are as follows.

THEOREM 3.1. *A stationary solution $u_0(x)$ is asymptotically stable if it satisfies $u_{0x}(x) \neq 0$ for all x in $[0, l)$ or for all x in $(0, l]$.*

THEOREM 3.2. *A stationary solution $u_0(x)$ is unstable if it satisfies $u_{0x}(x) = 0$ at two or more points x in $(0, l]$ or in $[0, l)$.*

Next, let us consider a stationary solution $u_0(x)$ which satisfies, for a certain x_1 lying in $(0, l)$,

$$\begin{aligned} u_{0x}(x) &> 0 \qquad \textit{for all } x \textit{ in} \quad [0, x_1), \\ u_{0x}(x_1) &= 0, \\ u_{0x}(x) &< 0 \qquad \textit{for all } x \textit{ in} \quad (x_1, l]. \end{aligned} \tag{3.6}$$

According to Theorem 2.2, there exists a stationary solution of this type if and only if the equation $S(D) = l$ has a real solution $D = D^*$ in the range (2.5). We have four theorems regarding the stability of this stationary solution.

THEOREM 3.3. *Let $u_0(x)$ be a stationary solution of the type* (3.6), *and D^* be a solution of $S(D) = l$. In this case,* (i) *$u_0(x)$ is asymptotically stable if $S'(D^*) > 0$, and* (ii) *$u_0(x)$ is unstable if $S'(D^*) < 0$.*

THEOREM 3.4. *Let $u_0(x)$ and D^* be as in Theorem* 3.3. *If $S'(D^*) = 0$ and $S''(D^*) > 0$ $[S''(D^*) < 0]$, then $u_0(x)$ is asymptotically stable with respect to positive [negative] disturbances, but not stable with respect to negative [positive] disturbances.*

THEOREM 3.5. *Let $u_0(x)$ and D^* be as in Theorem* 3.3. *If $S'(D^*) = S''(D^*) = 0$ and $S''(D^*) > 0$ $[S''(D^*) < 0]$, then $u_0(x)$ is asymptotically stable [unstable].*

THEOREM 3.6. *Let $u_0(x)$ and D^* be as in Theorem* 3.3. *If there exists a $\delta > 0$ such that $S'(D) = 0$ for all D in $(D^* - \delta, D^* + \delta)$, then $u_0(x)$ is neutrally stable.*

Next let us consider a stationary solution of the type (2.14). A stationary solution of this type exists if and ony if there exists a $D = D^*$ satisfying (2.15).

THEOREM 3.7. *Let $u_0(x)$ be a stationary solution of the type* (2.14), *and D^* be a solution of* (2.15). *In this case,* (i) *if $f(u_1) + f(u_2) < 0$ $[f(u_1) + f(u_2) > 0]$, then $u_0(x)$ is asymptotically stable with respect to positive [negative] disturbances, but* *not stable with respect to negative [positive] disturbances. Also,* (ii) *if* $f(u_1) + f(u_2) = 0$ *and* $T'(D^*) > 0$ $[T'(D^*) < 0]$, *then* $u_0(x)$ *is asymptotically stable [not stable].*

3.3. *Main Lemmas* (I)

Theorems 3.1–3.7 are proved by using Lemmas 3.1–3.5 which we shall formulate below.

Let $z(x; \mu)$ be a solution of the equation

$$z_{xx}(x; \mu) + f(z(x; \mu)) = 0 \tag{3.7a}$$

subject to the intial condition

$$z(0; \mu) = u_0(0) = u_1, \qquad z_x(0; \mu) = u_{0x}(0) + \mu. \tag{3.7b}$$

Clearly from (2.1) and (3.7), this function satisfies

$$z(x; 0) = u_0(x), \qquad 0 \leqslant x \leqslant l, \tag{3.8a}$$

$$z_\mu(0; \mu) = 0, \qquad z_{\mu x}(0; \mu) = 1. \tag{3.8b}$$

The following lemmas hold.

LEMMA 3.1. *If* $z_\mu(x; 0) > 0$ *for all* x *in* $(0, l]$, *then* $u_0(x)$ *is asymptotically stable.*

LEMMA 3.2. *If* $z_\mu(x; 0)$ *has a zero in* $(0, l)$, *then* $u_0(x)$ *is unstable.*

LEMMA 3.3. *If* $z(x; \mu)$ *satisfies*

$$\begin{aligned} &z_\mu(0; 0) = z_\mu(l; 0) = 0, \\ &z_\mu(x; 0) > 0 \qquad \text{for all } x \quad \text{in} \quad (0, l), \end{aligned} \tag{3.9}$$

and $z_{\mu\mu}(l; 0) > 0$ $[z_{\mu\mu}(l; 0) < 0]$, *then* $u_0(x)$ *is asymptotically stable with respect to positive [negative] disturbances, but not stable with respect to negative [positive] disturbances.*

LEMMA 3.4. *If* $z(x; \mu)$ *satisfies* (3.9), $z_{\mu\mu}(l; 0) = 0$, *and* $z_{\mu\mu\mu}(l; 0) > 0$ $[z_{\mu\mu\mu}(l; 0) < 0]$, *then* $u_0(x)$ *is asymptotically stable [unstable].*

LEMMA 3.5. *If* $z(x; \mu)$ *satisfies* (3.9) *and* $z_\mu(l; \mu) = 0$ *for all* μ *in some interval* $(-\delta_1, \delta_1)$, $\delta_1 > 0$, *then* $u_0(x)$ *is neutrally stable.*

These lemmas will be proved in Section 5.

3.4. *Main Lemmas* (II)

Four lemmas are given below. Theorems 3.1–3.7 are straightforward consequences of these lemmas and of Lemmas 3.1–3.5. The first is as follows.

LEMMA 3.6. (i) *If* $u_0(x)$ *satisfies the assumption of Theorem* 3.1, *then* $z_\mu(x; 0) > 0$ *for all* x *in* $(0, l]$.

(ii) *If* $u_0(x)$ *satisfies the assumption of Theorem* 3.2, *then* $z_\mu(x; 0)$ *has at least one zero in* $(0, l)$.

Theorems 3.1 and 3.2 are obtained by using Lemma 3.6 and 3.1–3.2. The second and the third lemma are as follows.

LEMMA 3.7. *Let* $u_0(x)$ *be as in Theorems* 3.3–3.6. *In this case* $z_\mu(x; 0)$ *has at most one zero in* $(0, l]$.

LEMMA 3.8. *Let* $u_0(x)$ *and* D^* *be as in Theorems* 3.3–3.6. *In this case*, (i) $z_\mu(l; 0) = -u_{0x}(0)\, u_{0x}(l)\, S'(D^*)$, (ii) $z_{\mu\mu}(l; 0) = -\{u_{0x}(0)\}^2\, u_{0x}(l)\, S''(D^*)$ *if* $S'(D^*) = 0$, (iii) $z_{\mu\mu\mu}(l; 0) = -\{u_{0x}(0)\}^3\, u_{0x}(l)\, S''(D^*)$ *if* $S'(D^*) = S''(D^*) = 0$, *and* (iv) $z_\mu(l; \mu) = 0$ *for all* μ *in some interval* $(-\delta_1, \delta_1)$, $\delta_1 > 0$, *if the equality* $S'(D) = 0$ *holds for all* D *in* $(D^* - \delta, D^* + \delta)$, $\delta > 0$.

In Theorems 3.3–3.6, $u_0(x)$ is assumed to satisfy (3.6). This assumption implies $u_{0x}(0) > 0$ and $u_{0x}(l) < 0$. Hence, according to Lemma 3.8(i), the sign of $z_\mu(l; 0)$ coincides with the sign of $S'(D^*)$. Therefore, according to Lemma 3.7, the inequality $z_\mu(x; 0) > 0$ holds for all x in $(0, l]$ if $S'(D^*) > 0$. Also, $z_\mu(x; 0)$ has one zero in $(0, l)$ if $S'(D^*) < 0$. Thus Theorem 3.3 follows from Lemmas 3.1 and 3.2. In the case $S'(D^*) = 0$, $z_\mu(x; 0)$ satisfies (3.9) on account of Lemma 3.8(i) and Lemma 3.7. Hence Theorems 3.4–3.6 follow immediately from Lemmas 3.8(ii)–(iv) and 3.3–3.5.

The fourth lemma is as follows.

LEMMA 3.9. *Let* $u_0(x)$ *and* D^* *be as in Theorem* 3.7. *In this case*, (i) $z_\mu(x; 0)$ *satisfies* (3.9). (ii) $z_{\mu\mu}(l; 0) = \{f(u_1) + f(u_2)\}\,\{f(u_1) - f(u_2)\}/\{f(u_1)\}^2 f(u_2)$ (iii) $z_{\mu\mu\mu}(l; 0) = T'(D^*)$ *if* $f(u_1) + f(u_2) = 0$.

In Theorems 3.7 and 3.8, $u_0(x)$ is assumed to satisfy (2.14). This assumption implies that

$$u_{0xx}(0) = -f(u_0(0)) = -f(u_1) > 0,$$
$$u_{0xx}(l) = -f(u_0(l)) = -f(u_2) < 0.$$

Hence, on account of Lemma 3.9(i), the sign of $z_{\mu\mu}(l; 0)$ coincides with the sign of $-\{f(u_1) + f(u_2)\}$. Thus Theorems 3.7 and 3.8 follow from Lemmas 3.4, 3.5, and 3.9.

Lemmas 3.6–3.9 are proved in Section 4.

4. Proofs of Lemmas 3.6–3.9

4.1. *Proof of Lemma* 3.6

The following lemma, which is a special case of the Sturm Comparison Theorem [1, Chap. 8, 1.1], plays an essential role in this section.

Lemma 4.1. *Let $g_1(x)$ and $g_2(x)$ be any linearly independent solutions of the equation*

$$g_{xx}(x) + f'(u_0(x))\, g(x) = 0. \tag{4.1}$$

If x_1 and x_2 lie in $[0, l]$ with $x_1 < x_2$, and if $g_1(x_1) = g_1(x_2) = 0, g_1(x) > 0$ for all x in (x_1, x_2), then $g_2(x)$ has exactly one zero in $[x_1, x_2]$.

The following equality is obtained by differentiating (2.1a) with respect to x:

$$\{u_{0x}(x)\}_{xx} + f(u_0(x))\, u_{0x}(x) = 0. \tag{4.2}$$

Hence $g_1(x) \equiv u_{0x}(x)$ is a nontrivial solution of (4.1). On the other hand, the following equalities are obtained by differentiating (3.7) with respect to μ, by setting $\mu = 0$ and by using (3.8a):

$$\{z_\mu(x; 0)\}_{xx} + f'(u_0(x))\, z_\mu(x; 0) = 0, \tag{4.3a}$$

$$z_\mu(0; 0) = 0, \qquad z_{\mu x}(0; 0) = 1. \tag{4.3b}$$

Hence $g_2(x) \equiv z_\mu(x; 0)$ is also a nontrivial solution of (4.1). By virtue of (4.3b), g_2 satisfies $g_2(0) = 0$ and $g_{2x}(0) = 1$.

In Lemma 3.6(i), $u_0(x)$ is assumed to satisfy $g_1(x) \equiv u_{0x}(x) \neq 0$ for all x in $[0, l)$ or for all x in $(0, l]$. In this case, according to Lemma 4.1, the inequality $g_2(x) \equiv z_\mu(x; 0) > 0$ holds for all x in $(0, l]$. On the other hand, in Lemma 3.6(ii), $g_1(x) \equiv u_{0x}(x)$ is assumed to have at least two zeros in $[0, l)$ or in $(0, l]$. In this case, according to Lemma 4.1, $g_2(x) \equiv z_\mu(x; 0)$ has a zero in $(0, l)$. Q.E.D.

4.2. *Proof of Lemma* 3.7

In this lemma, it is assumed that $g_1(x) \equiv u_{0x}(x)$ has exactly one zero in $(0, l)$. On the other hand, $g_2(x) \equiv z_\mu(x; 0)$ satisfies $g_2(0) = 0$, $g_{2x}(0) = 1$. Hence, according to Lemma 4.1, $g_2(x) \equiv z_\mu(x; 0)$ has at most one zero in $(0, l]$. Q.E.D.

4.3. *Proof of Lemma* 3.8

According to Theorem 2.2, a stationary solution $u_0(x)$ of type (3.6) is given by $u_0(x) = w(x; D^*)$, where w is a function defined by (2.3), and D^* is a solution of $S(D) = l$. Let us define a function $D(\mu)$ for $\mu \geqslant -u_{0x}(0)$ by

$$u_{0x}(0) + \mu = w_x(0; D(\mu)) \equiv (2\{D(\mu) - F(u_1)\}^{1/2}. \tag{4.5}$$

Clearly from the definition (3.7) of $z(x;\mu)$, the following equality holds:

$$z(x;\mu) = w(x; D(\mu)). \tag{4.6}$$

It is easily verified by the use of (2.3b), (4.5), and $w(x; D^*) = u_0(x)$ that

$$D(0) = D^*, \qquad D'(\mu) = w_x(0; D(\mu)). \tag{4.7}$$

Hence

$$z_\mu(l;\mu) = w_D(l; D(\mu))\, D'(\mu) = w_D(l; D(\mu))\, w_x(0; D(\mu)). \tag{4.8}$$

On the other hand, the following equality is obtained by differentiating (2.8) with respect to D:

$$w_x(S(D); D)\, S'(D) + w_D(S(D); D) = 0. \tag{4.9}$$

Hence, by using $S(D^*) = l$, we obtain $w_D(l; D^*) = w_x(l; D^*)\, S'(D^*) = u_{0x}(l)\, S'(D^*)$. Thus the equality $z_\mu(l; 0) = -u_{0x}(0)\, u_{0x}(l)\, S'(D^*)$ follows from (4.7) and (4.8).

Next let us consider the case $S'(D^*) = 0$. In this case, the following equality is obtained by differentiating (4.9) with respect to D and by using $S(D^*) = l$ and $w(x; D^*) = u_0(x)$:

$$u_{0x}(l)\, S''(D^*) + w_{DD}(l; D^*) = 0. \tag{4.10}$$

Hence (4.8) yields that

$$z_{\mu\mu}(l; 0) = w_{DD}(l; D^*)\, \{D'(0)\}^2 = -\{u_{0x}(0)\}^2\, u_{0x}(l)\, S''(D^*).$$

We can deduce the equality in Lemma 3.8(iii) in a similar way.

Finally let us consider the case where the equality $S'(D) = 0$ holds for all D in $(D^* - \delta, D^* + \delta)$. In this case, by virtue of $S(D^*) = l$, the equality $S(D) = l$ holds for all D in $(D^* - \delta, D^* + \delta)$. On account of $D(0) = D^*$, there exists a $\delta_1 > 0$ such that $D^* - \delta < D(\mu) < D^* + \delta$ for all μ in $(-\delta_1, \delta_1)$. Therefore, by virtue of (4.8) and (4.9), the equality $z_\mu(l;\mu) = 0$ holds for all μ in $(-\delta_1, \delta_1)$. Q.E.D.

4.4. *Proof of Lemma* 3.9

In this lemma, $u_0(x)$ is assumed to satisfy (2.14). In this case, by virtue of (4.2) and (4.3), $z_\mu(x; 0)$ is obtained explicitly as $z_\mu(x; 0) = u_{0x}(x)/u_{0xx}(0)$. Clearly this function satisfies (3.9).

A stationary solution $u_0(x)$ of type (2.14) is given by $u_0(x) = \hat{w}(x; D^*)$, where $\hat{w}$ is a solution of (2.13), and D^* is a solution of (2.15). We define functions $\bar{\theta}(\mu)$ and $\bar{D}(\mu)$ by

$$w(\bar{\theta}(\mu); \bar{D}(\mu)) = u_1, \quad w_x(\bar{\theta}(\mu); \bar{D}(\mu)) = \mu \ (= u_{0x}(0) + \mu).$$

On account of (3.7) and $u_0(x) = w(x; D^*)$, the following equalities hold:

$$\begin{aligned} z(x; \mu) &= w(x + \bar{\theta}(\mu); \bar{D}(\mu)), \\ \bar{\theta}(0) = 0, &\qquad \bar{D}(0) = D^*. \end{aligned} \tag{4.11}$$

The equalities in Lemma 3.9(ii), (iii) are obtained by using (2.13), (2.15), and (4.11). We omit details of the deduction. Q.E.D.

5. Stability Analysis—Proofs of Lemmas 3.1–3.5

5.1. *Preliminaries*

The following lemma holds regarding the stability of $u_0(x)$.

LEMMA 5.1. *A stationary solution $u_0(x)$ is stable if* (i) $z_\mu(x; 0) > 0$ *for all x in* $(0, l]$ *or if* (ii) $z(x; \mu)$ *satisfies* (3.9) *and* $z_{\mu\mu\mu}(l; 0) > 0$ *or if* (iii) $z(x; \mu)$ *satisfies* (3.9) *and* $z_\mu(l; \mu) = 0$ *for all* μ *in* $(-\delta_1, \delta_1)$, $\delta_1 > 0$. *Further,* $u_0(x)$ *is stable with respect to positive* [*negative*] *disturbances if* (iv) $z(x; \mu)$ *satisfies condition* (3.9) *and* $z_{\mu\mu}(l; 0) > 0$ $[z_{\mu\mu}(l; 0) < 0]$.

Proof. On account of (3.8), the following inequality holds for any given $\epsilon > 0$ if $\mu_1 > 0$ is sufficiently small.

$$\max_{0 \leqslant x \leqslant l}\{| z(x; \mu_1) - u_0(x)| + | z(x; -\mu_1) - u_0(x)|\} < \epsilon. \tag{5.1}$$

Let us consider the case where $z(x; \mu)$ satisfies the assumptions (i) or (ii) or (iii). If (i) or (ii) is satisfied, then the following inequalities hold when $\mu_1 > 0$ is sufficiently small.

$$z(x; \mu_1) > u_0(x) > z(x; -\mu_1) \qquad \textit{for all } x \textit{ in} \quad (0, l), \tag{5.2a}$$

$$z_x(0; \mu_1) > u_{0x}(0) > z_x(0; -\mu_1), \tag{5.2b}$$

$$z(l; \mu_1) > u_0(l) > z(l; -\mu_1). \tag{5.2c}$$

Further, if (iii) is satisfied, then $z(x; \mu)$ satisfies (5.2a, b), $z(l; \mu_1) = u_0(l) = z(l; -\mu_1)$ and $z_x(l; \mu_1) < u_{0x}(l) < z_x(l; -\mu_1)$. Hence, in these cases, the following inequalities hold if the disturbance $v_0(x)$ satisfies (3.1) and (3.3), and $\delta > 0$ in (3.3) is sufficiently small.

$$z(x; \mu_1) \geqslant u_0(x) + v_0(x) \geqslant z(x; -\mu_1) \qquad \textit{for all } x \textit{ in} \quad [0, l].$$

Therefore, according to the Comparison Theorem with respect to parabolic equations (see [3, Chap. 2]), the solution $u(x, t)$ subject to the initial condition $u(x, 0) = u_0(x) + v_0(x)$ satisfies $z(x; \mu_1) \geqslant u(x, t) \geqslant z(x; -\mu_1)$ for all (x, t) in $[0,] \times [0, \infty)$. Thus, by using (5.1), it is shown that $u_0(x)$ is stable.

Next let us consider the case where the assumption (iv) is satisfied. In this case, $z(x; \mu)$ satisfies (5.2b) and $z(x; \mu_1) > u_0(x)$ $[u_0(x) > z(x; -\mu_1)]$ for all x in $(0, l]$ if $\mu_0 > 0$ is sufficiently small. Hence, if $v_0(x)$ satisfies (3.1), (3.3), and $v_0(x) \geqslant 0$ $[v_0(x) \leqslant 0]$, and if $\delta > 0$ in (3.3) is sufficiently small, then the following inequalities hold.

$$z(x; \mu_1) \geqslant u_0(x) + v_0(x) \geqslant u_0(x),$$
$$[u_0(x) \geqslant u_0(x) + v_0(x) \geqslant z(x; -\mu_1)].$$

Therefore the solution $u(x, t)$ subject to the initial condition $u(x, 0) = u_0(x) + v_0(x)$ satisfies the inequalities $z(x; \mu_1) \geqslant u(x, t) \geqslant u_0(x)$ $[u_0(x) \geqslant u(x, t) \geqslant z(x; -\mu_1)]$ for all (x, t) in $[0, l] \times [0, \infty)$. Thus it is shown that $u_0(x)$ is stable with respect to positive [negative] disturbances. Q.E.D.

Consider the eigenvalue problem

$$\lambda\varphi(x) = \varphi_{xx}(x) + f'(u_0(x))\, \varphi(x), \qquad 0 < x < l, \tag{5.3a}$$

$$\varphi(0) = \varphi(l) = 0. \tag{5.3b}$$

According to Sturm–Liouville theory (see [1, Chap. 8]), Eqs. (5.3) yields countably many eigenvalues λ_k, $k \geqslant 0$, and corresponding eigenfunctions $\varphi_k(x)$, $k \geqslant 0$. All the eigenvalues are real and $\lambda_k \to -\infty$ as $k \to +\infty$. Without loss of generality we may assume that $\lambda_0 > \lambda_1 > \lambda_2 > \cdots$. The eigenfunctions φ_k can be chosen so that $\int_0^l \varphi_i(x)\, \varphi_j(x)\, dx = \delta_{ij}$. Some further properties of these eigenfunctions are stated in the following lemma, which is a special case of Theorem 2.8 in [1, Chap. 8].

LEMMA 5.2. *For each integer $k \geqslant 0$ the eigenfunction φ_k has exactly k zeros in* $(0, l)$. *In particular, $\varphi_0(x)$ has no zeros in* $(0, l)$.

The following lemma is an immediate consequence of the variational principle (see [2, Chap. 4].

LEMMA 5.3. (i) *Let $g(x)$ be any twice continuously differentiable function on* $[0, l]$ *satisfying $g(0) = g(l) = 0$. Then*

$$\int_0^l g(x)\, \{g_{xx}(x) + f'(u_0(x))\, g(x)\}\, dx \leqslant \lambda_0 \int_0^l \{g(x)\}^2\, dx.$$

(ii) *If $g(x)$ satisfies $\int_0^l g(x)\, \varphi_0(x)\, dx = 0$ in addition to $g(0) = g(l) = 0$, then*

$$\int_0^l g(x)\, \{g_{xx}(x) + f'(u_0(x))\, g(x)\}\, dx \leqslant \lambda_0 \int_0^l \{g(x)\}^2\, dx.$$

It follows from (4.3) that $z_\mu(x; 0)$ is a nontrivial solution of Eq. (5.3a) with $\lambda = 0$. Hence the following lemma is obtained as an immediate consequence of Lemma 5.2.

LEMMA 5.4. (i) *If $z_\mu(x; 0)$ satisfies* (3.9), *then* $\lambda_0 = 0$ *and* $\varphi_0(x) = c z_\mu(x; 0)$, *where c is a normalizing factor.*

Further, the following lemma is obtained by using the Sturm Comparison Theorem (see Theorem 1.1 in [1, Chap. 8]) and the latter part of Lemma 5.2.

LEMMA 5.4. (ii) *If $z_\mu(x; 0) > 0$ for all x in* $(0, l]$, *then* $\lambda_0 < 0$.

(iii) *If $z_\mu(x; 0)$ has a zero in* $(0, l)$, *then* $\lambda_0 > 0$.

Let $v(x, t)$ denote a function defined by

$$v(x, t) \equiv u(x, t) - u_0(x),$$

where $u(x, t)$ is a solution of (1.1) subject to the initial condition (3.2). Clearly $v(x, t)$ satisfies the following equations.

$$v_t(x, t) = v_{xx}(x, t) + f(u_0(x) + v(x, t)) - f(u_0(x)), \tag{5.4a}$$

$$v(0, t) = v(l, t) = 0, \qquad t \geqslant 0, \tag{5.4b}$$

$$v(x, 0) = v_0(x), \qquad 0 \leqslant x \leqslant l. \tag{5.4c}$$

The following lemma is easily proved by using mathematical tools developed in [3].

LEMMA 5.5. *If a solution $v(x, t)$ of the parabolic equation* (5.4) *satisfies* $\int_0^\infty \{\int_0^l \{v(x, t)\}^2\, dx\}\, dt < +\infty$, *then* $\lim_{t\to\infty} \mathrm{Max}_{0\leqslant x\leqslant l} | v(x, t)| = 0$.

5.2. *Proof of Lemma* 3.1

Equation (5.4a) can be rewritten as follows by expanding the term $f(u_0 + v)$ in Taylor series:

$$v_t = v_{xx} + f'(u_0)\, v + Gv^2, \tag{5.5}$$

where $G \equiv \frac{1}{2} f''(u_0 + \theta v)$, and $\theta(x, t)$ is a function satisfying $0 \leqslant \theta(x, t) \leqslant 1$. Multiplying this by $v(x, t)$, and integrating on $[0, l]$ with respect to x, we obtain

$$\begin{aligned}\int_0^l v v_t\, dx &= \int_0^l v\{v_{xx} + f'(u_0)\, v\}\, dx + \int_0^l Gv^3\, dx \\ &\leqslant \int_0^l \{\lambda_0 + Gv\}\, v^2\, dx.\end{aligned}$$

In Lemma 3.1, $z(x; \mu)$ is assumed to satisfy $z_\mu(x; 0) > 0$ for all x in $(0, l]$. Hence, on account of Lemma 5.4(ii), λ_0 is negative. Also, on account of Lemma 5.1, the inequality $\frac{1}{2} | \lambda_0 | > \mathrm{Max}_{0\leqslant x\leqslant l, t\geqslant 0} | G(x, t)\, v(x, t)|$ holds if $\delta > 0$ in (3.3) is sufficiently small. Hence the following inequality is obtained.

$$\frac{1}{2} \frac{d}{dt} \int_0^l \{v(x, t)\}^2\, dx \leqslant \frac{1}{2} \lambda_0 \int_0^l \{v(x, t)\}^2\, dx.$$

Thus, $\int_0^l \{v(x, t)\}^2\, dx \leqslant \exp(\lambda_0 t) \int_0^l \{v_0(x)\}^2\, dx$. Therefore, by using Lemma 5.5 and $\lambda_0 < 0$, we obtain the asymptotic stability of $u_0(x)$. Q.E.D.

5.3. *Proof of Lemma* 3.2

Let $\alpha(t)$ and $\bar{v}(x, t)$ denote functions defined by

$$\alpha(t) \equiv \int_0^l v(x, t)\, \varphi_0(x)\, dx, \tag{5.6a}$$

$$\bar{v}(x, t) \equiv v(x, t) - \alpha(t)\, \varphi_0(x). \tag{5.6b}$$

Clearly $\bar{v}(x, t)$ satisfies

$$\int_0^l \bar{v}(x, t)\, \varphi_0(x)\, dx = 0, \qquad \bar{v}(0, t) = \bar{v}(l, t) = 0. \tag{5.7}$$

Hence, by using Lemma 5.3(ii), we obtain

$$\int_0^l \bar{v}\{\bar{v}_{xx} + f'(u_0)\, \bar{v}\}\, dx \leqslant \lambda_1 \int_0^l \bar{v}^2\, dx. \tag{5.8}$$

Substituting (5.6b) in (5.5), we obtain

$$\begin{aligned} \alpha_t \varphi_0 + \bar{v}_t &= \{\varphi_{0xx} + f'(u_0)\, \varphi_0\}\, \alpha + \{\bar{v}_{xx} + f'(u_0)\, \bar{v}\} + Gv^2 \\ &= \lambda_0 \varphi_0 \alpha + \{\bar{v}_{xx} + f'(u_0)\, \bar{v}\} + (\alpha\varphi_0 + \bar{v})\, Gv. \end{aligned} \tag{5.9}$$

Multiplying this by $\alpha\varphi_0$, and taking account of (5.7a) and

$$\int_0^l \varphi_0\{\bar{v}_{xx} + f'(u_0)\, \bar{v}\}\, dx = \int_0^l \bar{v}\{\varphi_{0xx} + f'(u_0)\, \varphi_0\}\, dx = \lambda_0 \int_0^l \bar{v}\varphi_0\, dx = 0,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\alpha^2 = \left\{\lambda_0 + \int_0^l Gv\varphi_0{}^2\, dx\right\} \alpha^2 + \alpha \int_0^l Gv\bar{v}\varphi_0\, dx. \tag{5.10a}$$

Similarly, multiplying (5.9) by $\bar{v}(x, t)$, integrating on $[0, l]$, and taking account of (5.7) and (5.8), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^l \bar{v}^2\, dx \leqslant \int_0^l \{\lambda_1 + Gv\}\, \bar{v}^2\, dx + \alpha \int_0^l Gv\bar{v}\varphi_0\, dx. \tag{5.10b}$$

In Lemma 3.2, $z_\mu(x; 0)$ is assumed to have a zero in $(0, l)$. In this case, Lemma 5.4(iii) yields that $\lambda_0 > 0$. Let us prove the instability of $u_0(x)$ by contradiction. We suppose that $u_0(x)$ is stable, i.e., that the inequality $|\, v(x, t)| < \epsilon$ holds for all $t \geqslant 0$. Let A^2 be any positive constant lying in the interval (λ_1, λ_0). If $\epsilon > 0$ is sufficiently small, then the following inequality holds for all (x, t) in $[0, l] \times [0, \infty)$:

$$\lambda_0 + \int_0^l Gv\varphi_0{}^2\, dx > A^2 > \lambda_1 + Gv.$$

By using this and (5.10), we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\alpha^2 - \int_0^l \bar{v}^2\,dx\right\} \geqslant A^2\left\{\alpha^2 - \int_0^l \bar{v}^2\,dx\right\}. \tag{5.11}$$

Let us choose the initial data $v(x, 0) = v_0(x)$ so that $\alpha(0) \neq 0$, $\bar{v}(x, 0) = 0$. In this case, (5.11) yields $\{\alpha(t)\}^2 \geqslant \{\alpha(0)\}^2 \exp(2A^2t)$. Clearly this inequality contradicts the supposition that $u_0(x)$ is stable. Q.E.D.

5.4. *Proof of Lemma* 3.3

In Lemma 3.3, $z(x; \mu)$ is assumed to satisfy (3.9) and $z_{\mu\mu}(l; 0) \neq 0$. In this case, according to Lemma 5.4(i),

$$\lambda_0 = 0, \qquad \varphi_0(x) = cz_\mu(x; 0). \tag{5.12}$$

We may assume $c > 0$ without losing any generality. According to Lemma 5.1, $u_0(x)$ is stable with respect to positive [negative] disturbances if $z_{\mu\mu}(l; 0) > 0$ [if $z_{\mu\mu}(l; 0) < 0$]. In the following, we shall consider the case $z_{\mu\mu}(l; 0) > 0$ only.

The following equalities are obtained by differentiating (3.7) twice with respect to μ, and by using $z(x; 0) = u_0(x)$ and (5.12).

$$\begin{aligned} \{z_{\mu\mu}(x; 0)\}_{xx} + f'(u_0(x))\, z_{\mu\mu}(x; 0) + f''(u_0(x))\,\{\varphi_0(x)/c\}^2 &= 0, \\ z_{\mu\mu}(0; 0) &= 0. \end{aligned} \tag{5.13}$$

Equation (5.4a) can be rewritten as follows by expanding the term $f(u_0 + v)$ in a Taylor series:

$$v_t = v_{xx} + f'(u_0)\, v + \tfrac{1}{2}f''(u_0)\, v^2 + Hv^3, \tag{5.14}$$

where $H \equiv \frac{1}{6}f^{(3)}(u_0 + \bar{\theta}v)$ and $0 \leqslant \bar{\theta}(x, t) \leqslant 1$. Let $\alpha(t)$ and $\bar{v}(x, t)$ be as in (5.6). Substituting (5.6b) in (5.14) and taking account of (5.12) and (5.13), we obtain

$$\begin{aligned} \alpha_t\varphi_0 + \bar{v}_t &= \bar{v}_{xx} + f'(u_0)\bar{v} + \{\varphi_{0xx} + f'(u_0)\varphi_0\}\,\alpha + \tfrac{1}{2}f''(u_0)(\alpha\varphi_0)^2 + P \\ &= \bar{v}_{xx} + f'(u_0)\,\bar{v} - \tfrac{1}{2}c^2\{(z_{\mu\mu})_{xx} + f'(u_0)z_{\mu\mu}\}\,\alpha^2 + P, \end{aligned}$$

where $P \equiv \frac{1}{2}f''(u_0)\,\{2\alpha\varphi_0\bar{v} + \bar{v}^2\} + Hv^3$ and $z_{\mu\mu} \equiv z_{\mu\mu}(x; 0)$.

The following equalities hold:

$$\int_0^l \{\bar{v}_{xx} + f'(u_0)\,\bar{v}\}\,\varphi_0\,dx = \int_0^l \bar{v}\{\varphi_{0xx} + f'(u_0)\,\varphi_0\}\,dx = 0,$$

$$\begin{aligned} &\int_0^l \{z_{\mu\mu xx} + f'(u_0)\,z_{\mu\mu}\}\,\varphi_0\,dx \\ &\quad = \int_0^l [\{z_{\mu\mu x}\varphi_0 - z_{\mu\mu}\varphi_{0x}\}_x + z_{\mu\mu}\{\varphi_{0xx} + f'(u_0)\,\varphi_0\}]\,dx \\ &\quad = -z_{\mu\mu}(l; 0)\,\varphi_{0x}(l; 0). \end{aligned}$$

Hence, multiplying (5.14) by $\varphi_0(x)$, integrating on $[0, l]$ and taking account of (5.7) and the above equalities, we obtain

$$\alpha_t = -\tfrac{1}{2}c^3\alpha^2 B + \int_0^l P\varphi_0\, dx, \tag{5.15}$$

where $B \equiv -(1/c)\, z_{\mu\mu}(l; 0)\, \varphi_{0x}(l) = -z_{\mu\mu}(l; 0)\, z_{\mu x}(l; 0)$. By virtue of assumptions (3.9) and $z_{\mu\mu}(l; 0) > 0$, B is positive.

Let us assume that the initial data $v(x, 0) = v_0(x)$ for Eqs. (5.4) satisfies $v_0(x) \geqslant 0$. Since $u_0(x)$ is stable with respect to positive disturbance, the solution $v(x, t)$ of (5.4) satisfies the following inequality for any given $\epsilon > 0$ if $\delta > 0$ in (3.3) is sufficiently small:

$$\epsilon > v(x, t) \geqslant 0 \qquad \textit{for all } (x, t) \textit{ in } \quad [0, l] \times [0, \infty). \tag{5.16a}$$

By using (5.6a), (5.16a), and $\varphi_0(x) \geqslant 0$ we obtain

$$\epsilon \int_0^l \varphi_0(x)\, dx > \alpha(t) \geqslant 0. \tag{5.16b}$$

The following inequality holds for any $D^2 > 0$ on account of (5.10a) and (5.15).

$$\frac{d}{dt}\left\{\alpha + \frac{1}{2} D^2 \int_0^l \bar{v}^2\, dx\right\} \leqslant \int_0^l \{P_1(x, t) + P_2(x, t)\}\, dx,$$

where

$$P_1 \equiv -\tfrac{1}{2}c^3B(\alpha\varphi_0)^2 + f''(u_0)\, \varphi_0(\alpha\varphi_0)\bar{v} + \{\lambda_1 D^2 + \tfrac{1}{2}f''(u_0)\varphi_0\}\bar{v}^2,$$
$$P_2 \equiv \{H\varphi_0 v + D^2\, G\bar{v}\}v^2.$$

By virtue of $B > 0$ and $\lambda_1 < \lambda_0 = 0$, the following inequality holds if D^2 is sufficiently large.

$$\int_0^l P_1(x, t)\, dx \leqslant -\tfrac{1}{4}c^3B \int_0^l \{(\alpha\varphi_0)^2 + \bar{v}^2\}\, dx = -\tfrac{1}{4}c^3B \int_0^l v^2\, dx.$$

Hence, if $\epsilon > 0$ in (5.16a) is sufficiently small, then

$$\int_0^l (P_1 + P_2)\, dx \leqslant \int_0^l \{-\tfrac{1}{4}c^3B + H\varphi_0 v + D^2G\bar{v}\}\, v^2\, dx$$
$$\leqslant -\tfrac{1}{8}c^3B \int_0^l v^2\, dx.$$

Thus we obtain

$$\frac{d}{dt}\left\{\alpha + \frac{1}{2} D^2 \int_0^l \bar{v}^2\, dx\right\} \leqslant -\frac{1}{8}\, c^3B \int_0^l v^2\, dx. \tag{5.17}$$

Therefore, by virtue of $c^3B > 0$ and (5.16b), $v(x, t)$ must satisfy

$$\int_0^{\infty} \left[\int_0^l \{v(x, t)\}^2\, dx\right] dt < +\infty.$$

Hence, according to Lemma 5.5, $\lim_{t\to\infty} \operatorname{Max}_{0\leqslant x\leqslant l} | v(x, t)| = 0$. Thus the former part of Lemma 3.3 is proved.

Let us prove the latter part of Lemma 3.3 by contradiction. We suppose that $u_0(x)$ is stable with respect to negative disturbances, i.e., that $v(x, t)$ satisfies

$$0 \geqslant v(x, t) > -\epsilon \tag{5.18}$$

for all (x, t) in $[0, l] \times [0, \infty)$ if the initial data satisfies $v(x, 0) = v_0(x) \leqslant 0$ and if $\delta > 0$ in (3.3) is sufficiently small. Since (4.17) holds if $\epsilon > 0$ is sufficiently small, we obtain

$$\frac{d}{dt}\left\{\alpha + \frac{1}{2} D^2 \int_0^l \bar{v}^2\, dx\right\} \leqslant -\frac{1}{8} c^3B \int_0^l (\alpha\varphi_0 + \bar{v})^2\, dx \leqslant -\frac{1}{8} c^3B\alpha^2. \tag{5.19}$$

Let us choose the initial data $v(x, 0) = v_0(x)$ so that $\alpha(0) < 0$ and $\bar{v}(x, 0) = 0$. In this case, (5.19) yields that $\alpha(t) \leqslant -\frac{1}{8}c^3B\{\alpha(0)\}^2t$. Clearly this inequality contradicts the supposition (5.18). Thus it is proved that $u_0(x)$ is unstable with respect to negative disturbances. Q.E.D.

We can prove Lemma 3.4 in a similar way. We omits its proof.

5.5. *Proof of Lemma* 3.5

In this lemma, $z(x; \mu)$ is assumed to satisfy (3.9) and

$$z_\mu(0; \mu) = z_\mu(l; \mu) = 0 \qquad \text{for all } \mu \text{ in } \quad (-\delta_1, \delta_1). \tag{5.20}$$

In this case, according to Lemma 5.1, the solution $u(x, t)$ of Eqs. (1.1) with the initial condition (3.2) satisfies the following inequality for any given $\epsilon > 0$ if $\delta > 0$ in (3.3) is sufficiently small.

$$| u(x, t) - u_0(x)| < \epsilon \qquad \text{for all } (x, t) \text{ in } \quad [0, l] \times [0, \infty). \tag{5.21}$$

Consider the eigenvalue problem

$$\lambda\varphi(x) = \varphi_{xx}(x) + f'(z(x; \mu))\, \varphi(x), \qquad 0 < x < l, \tag{5.22a}$$

$$\varphi(0) = \varphi(l) = 0, \tag{5.22b}$$

where μ is a constant lying in $(-\delta_1, \delta_1)$. Equations (5.22) yield countably many real eigenvalues $\lambda_k(\mu)$, $k \geqslant 0$, and corresponding eigenfunctions $\varphi_k(x; \mu)$. We may assume $\lambda_0(\mu) > \lambda_1(\mu) > \cdots$ without losing any generality. In correspondence with Lemmas 5.2 and 5.3, the following lemma holds.

LEMMA 5.6. (i) *$\varphi_0(x;\mu)$ has no zeros in $(0, l)$.*

(ii) *If $g(x)$ is a twice continuously differentiable function satisfying $g(0) = g(l) = 0$ and $\int_0^l g(x)\,\varphi_0(x;\mu)\,dx = 0$, then*

$$\int_0^l g(x)\,\{g_{xx}(x) + f'(z(x;\mu))\,g(x)\}\,dx \leqslant \lambda_1(\mu)\int_0^l \{g(x)\}^2\,dx.$$

The following equality is obtained by differentiating (3.7a) with respect to μ.

$$\{z_\mu(x;\mu)\}_{xx} + f'(z(x;\mu))\,z_\mu(x;\mu) = 0. \tag{5.23}$$

Hence, taking account of (5.120), (3.9), and Lemma 5.6(i), we obtain

$$\lambda_0(\mu) = 0, \qquad \varphi_0(x;\mu) = c_1 z_\mu(x;\mu), \tag{5.24}$$

where c_1 is a normalizing factor.

Let $\bar\mu(t)$ denote the value of parameter μ which minimizes the following integral.

$$P[u;\mu] \equiv \tfrac{1}{2}\int_0^l \{u(x,t) - z(x;\mu)\}^2\,dx.$$

Clearly $\bar\mu(t)$ satisfies

$$P_\mu[u;\bar\mu] = -\int_0^l \hat v(x,t)\,z_\mu(x;\bar\mu(t))\,dx = 0, \tag{5.25}$$

where $\hat v(x,t) \equiv u(x,t) - z(x;\bar\mu(t))$. Further, on account of (5.21)) and $z(x;0) = u_0(x)$, $\bar\mu(t)$ lies in the interval $(-\delta_1, \delta_1)$ for all $t \geqslant 0$ if $\epsilon > 0$ in (5.21) is sufficiently small. In this case, $\hat v(x,t)$ satisfies the following boundary condition:

$$\hat v(0,t) = \hat v(l,t) = 0 \qquad \textit{for all } t \geqslant 0. \tag{5.26}$$

Thus, taking account of (5.24), (5.25), (5.26), and Lemma 5.6(ii), we obtain

$$\int_0^l \hat v(x,t)\,\{\hat v_{xx}(x,t) + f'(z(x;\bar\mu(t)))\,\hat v(x,t)\}\,dx \leqslant \lambda_1(\bar\mu(t))\int_0^l \{\hat v(x,t)\}^2\,dx. \tag{5.27}$$

The function $\hat v(x,t) \equiv u(x,t) - z(x;\bar\mu(t))$ satisfies the following equation.

$$\begin{aligned}
&\hat v_t(x,t) + \bar\mu_t z_\mu(x;\bar\mu)\\
&\quad = \hat v_{xx}(x,t) + f(z(x;\bar\mu) + \hat v(x,t)) - f(z(x;\bar\mu))\\
&\quad = \hat v_{xx}(x,t) + f'(z(x;\bar\mu))\,\hat v(x,t) + \hat G(x,t)\,\{\hat v(x,t)\}^2,
\end{aligned} \tag{5.28}$$

where $\hat G = \frac{1}{2}f''(z + \hat\theta\hat v)$ and $0 \leqslant \hat\theta(x,t) \leqslant 1$. Multiplying this by $\hat v(x,t)$, integrating on $[0, l]$ and taking account of (5.25) and (5.27), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^l \{\hat v(x,t)\}^2\,dx &\leqslant \int_0^l \{\lambda_1(\bar\mu) + G\hat v\}\,\hat v^2\,dx\\
&\leqslant \int_0^l \{-a^2 + G\hat v\}\,\hat v^2\,dx,
\end{aligned}$$

where $\operatorname{Max}_{-\delta_1 \leqslant \mu \leqslant \delta_1} \lambda_1(\mu) \equiv -a^2 < \lambda_0(\mu) = 0$. Hence the following inequality holds if $\epsilon > 0$ in (5.21) is sufficiently small.

$$\frac{1}{2} \frac{d}{dt} \int_0^l \{\hat{v}(x, t)\}^2 \, dx \leqslant -\frac{1}{2} a^2 \int_0^l \{\hat{v}(x, t)\}^2 \, dx.$$

This implies that

$$\int_0^l \{\hat{v}(x, t)\}^2 \, dx \leqslant \int_0^l \{\hat{v}(x, 0)\}^2 \, dx \exp(-a^2 t). \tag{5.29}$$

The following equality is obtained by differentiating (5.25) with respect to t.

$$\int_0^l \hat{v}_t(x, t) \, z_\mu(x; \bar{\mu}) \, dx + \bar{\mu}_t \int_0^l \hat{v}(x, t) \, z_{\mu\mu}(x; \bar{\mu}) \, dx = 0.$$

Further, the following equality is obtained by using (5.23) and (5.26).

$$\int_0^l \{\hat{v}_{xx}(x, t) + f'(z(x; \bar{\mu})) \, \hat{v}(x, t)\} \, z_\mu(x; \bar{\mu}) \, dx = 0.$$

Hence, multiplying (5.28) by $z_\mu(x; \bar{\mu})$, integrating on $[0, l]$ and taking account of these equalities we obtain

$$\bar{\mu}_t \int_0^l [\{z_\mu(x; \bar{\mu})\}^2 - \hat{v}(x, t) \, z_{\mu\mu}(x; \bar{\mu})] \, dx = \int_0^l G(x, t) \, \{\hat{v}(x, t)\}^2 \, z_\mu(x; \bar{\mu}) \, dx. \tag{5.30}$$

If $\epsilon > 0$ in (5.21) is sufficiently small, then

$$\int_0^l [\{z_\mu(x; \bar{\mu})\}^2 - \hat{v}(x, t) \, z_{\mu\mu}(x; \bar{\mu})] \, dx \geqslant A_1^2 > 0,$$

$$\int_0^l G(x, t) \, \{\hat{v}(x, t)\}^2 \, z_\mu(x; \bar{\mu}) \, dx \leqslant A_2^2 \exp(-a^2 t).$$

In this case, (5.30) implies $| \bar{\mu}_t | \leqslant A_2^2 \exp(-a^2 t)/A_1^2$. This implies that $\bar{\mu}(t)$ converges to a certain point as $t \to \infty$. Further (5.29) and Lemma 5.5 imply that $\lim_{t \to \infty} \operatorname{Max}_{0 \leqslant x \leqslant l} | \hat{v}(x, t)| = 0$. Thus it is shown that $u(x, t) \equiv z(x; \bar{\mu}(t)) + \hat{v}(x, t)$ satisfies $\lim_{t \to \infty} u(x, t) = z(x; \mu^*)$, where μ^* is a certain point lying in $(-\delta_1, \delta_1)$. Q.E.D.

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